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Unified Literal Approximations for Longitudinal Dynamics of Flexible Flight Vehicles

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The literal approximation method has been recently used to obtain simple literal (analytical) approximations for key longitudinal dynamic characteristics of a given large, flexible aircraft. These approximations constitute a useful additional tool to existing design and simulation procedures for the particular flexible flight vehicle for which they were derived. A desirable feature of this tool is generality, i.e., the suitability of one set of unified literal approximations to adequately describe the dynamics of significantly different flexible flight vehicles. This problem is addressed by incorporating symbolic manipulation software to facilitate both the ease of use and the derivation accuracy of the otherwise algebraically intensive literal approximation method used. The resulting automated literal method is used to independently derive literal approximations for the longitudinal dynamics of three significantly different flexible flight vehicle configurations, namely, a large aircraft, a large supersonic missile, and a hypersonic vehicle. The unified set of literal approximations is then synthesized from these three sets of independently derived literal approximations, yielding a new, unified formulation of the flexibility effects on the poles and zeros of a general flexible flight vehicle. The unified approximations lead to marked improvement in modeling accuracy over the traditional approximations (derived using the decoupled rigid body and elastic modes). These improvements are demonstrated using detailed tables and an extensive set of frequency response plots.

Nomenclature

F_{1_α} = dimensional generalized force in η_1, per α, 1/s^2
F_{1_q} = dimensional generalized force in η_1, per q, 1/s
F_{1_δ} = dimensional generalized force in η_1, per δ, 1/s^2
$F_{1_{\eta 1}}$ = dimensional generalized force in η_1, per η_1, 1/s^2
$F_{1_{\dot\eta 1}}$ = dimensional generalized force in η_1, per $\dot\eta_1$, 1/s
M_q = dimensional pitching moment per q, 1/s
M_α = dimensional pitching moment per α, 1/s^2
M_δ = dimensional pitching moment per δ, 1/s^2
M_{η_1} = dimensional pitching moment per η_1, 1/s^2
$M_{\dot\eta_1}$ = dimensional pitching moment per $\dot\eta_1$, 1/s
q = pitch rate, rad/s
s = Laplace transform variable, 1/s
V_T = flight vehicle airspeed, ft/s
Z_q = dimensional vertical acceleration per q, ft/s
Z_α = dimensional vertical acceleration per α, ft/s^2
Z_δ = dimensional vertical acceleration per δ, ft/s^2
Z_{η_1} = dimensional vertical acceleration per η_1, ft/s^2
$Z_{\dot\eta_1}$ = dimensional vertical acceleration per $\dot\eta_1$, ft/s
α = angle of attack, rad
δ = deflection angle of pitch control surface, rad
ζ_1 = in vacuo damping for first aeroelastic mode
η_1 = generalized coordinate for first aeroelastic mode
$\dot\eta_1$ = rate of generalized coordinate for first aeroelastic mode, 1/s
θ = pitch angle, rad
ω_1 = in vacuo natural frequency of first aeroelastic mode, rad/s

Introduction

TO aid in the dynamic analysis and control system synthesis of flexible flight vehicles, simple reduced-order literal models of the vehicle's dynamics are sought. Such models are useful for identifying the principal physical design factors (e.g., certain geometric, structural, and aerodynamic parameters) affecting the vehicle dynamics and for linking them with the design of the controller and the resulting closed-loop properties of the system.[1,2] The reduced-order literal models should provide a good representation of the critical characteristics relevant to the design of a feedback system and help to expose the underlying physical causes for critical dynamic characteristics. Since exact literal expressions become cumbersome for the roots of polynomial of order $n \geq 3$ and are generally nonexistent for $n \geq 5$, one is faced with two options: 1) compromise the literal nature of the solution in favor of numerical accuracy, i.e., use strictly numerical methods, or 2) compromise the numerical accuracy of the literal expressions when dealing with higher order literal models and scale down the analysis to seeking only approximate literal expressions for the critical dynamic parameters of interest. Following option 2 and extending the work of McRuer et al.[1] and Pearce et al.,[3] Newman and Schmidt[2] established an algebraic method to obtain approximate literal expressions for the poles and zeros of each element in the transfer function of a specific large, flexible aircraft configuration.[4,5] Using a first-order sensitivity approach, the coefficients of each numerator or denominator polynomial were approximated by a first-order Taylor series expansion about the coefficients of a nominal polynomial prespecified by the user. The resulting approximations were then used to obtain a literal expansion of the actual polynomial roots about the known literal expressions for the roots of the nominal polynomial, yielding a literal approximation for the poles and zeros of each element of the system transfer function matrix.

Manual application of the method to a fifth-order model for the longitudinal dynamics of a flexible aircraft[2] proved very time consuming. Therefore, a key concern regarding the literal approximation method is its usefulness as an additional tool for existing design and simulation procedures. The method would be considered useful if it is easy to apply and yields simple literal expressions that adequately describe the dynamics of the flexible flight vehicle analyzed. Generality and, consequently, further simplification of the literal approximation method would be enhanced if one can demonstrate that a single set of simple literal expressions can adequately describe the dynamics of a large variety of significantly different flexible flight vehicles.

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Table 1 Flight conditions and mass properties of flight vehicles analyzed

	Flight conditions		Mass properties	
Flight vehicle	Altitude, ft	Mach number	Weight, lb	Pitch inertia, slug ft^2
Flexible Aircraft[a]	5,000	0.6	288,017	6.400×10^6
Supersonic missile[b]	0	1.84	300,000	5.833×10^6
Hypersonic vehicle[c]	85,000	8.00	300,000	18.650×10^6

[a]From Refs. 2, 4. [b]From Ref. 3. [c]From Ref. 6.

To develop results for several vehicle configurations and to further explore the literal approximation method, we attempted to obtain one set of unified approximations for the longitudinal dynamics of three significantly different flexible flight vehicles. Literal approximations were derived independently for each one of these vehicle models, and a single set of unified literal approximation was then formed from their union. The following sections describe 1) the flexible flight vehicles analyzed and their longitudinal dynamics model, 2) the synthesis of the literal approximation method with the Theorist®[6] symbolic manipulation software, and 3) the unified set of approximations obtained and its resulting accuracy.

Flexible Vehicles Analyzed and Their Longitudinal Model

The equations of motion for a generic flexible vehicle were derived about its mean axis coordinate system using Lagrange equation and the principle of virtual work.[4] The elastic deformation of the vehicle was modeled using mode shape functions and in vacuo modal vibration frequencies obtained from the structural dynamic analyses.[3–5] The three flexible flight vehicle configurations analyzed are a large flexible aircraft,[2,4] a supersonic missile,[3] and a hypersonic vehicle.[7] These configurations are shown in Fig. 1. The aircraft has a reasonably conventional geometry with a low aspect ratio, swept wing, conventional tail for pitch control, and a small canard for structural mode control. The missile has a large canard for pitch control and a low-aspect-ratio wing. The hypersonic vehicle is a statically unstable lifting body with trailing-edge pitch control surfaces. The flight conditions and the mass properties of these configuration are summarized in Table 1, and the values of their stability derivatives are listed in Table 2.

The complete linearized formulation for the longitudinal dynamics of generic flexible flight vehicle in a steady-state level flight can be found in Pearce et al.,[3,4] for example. Newman and Schmidt[2] obtained the short-period approximation for the large flexible aircraft of Fig. 1 by neglecting the equation and degrees of freedom associated with its forward velocity degree of freedom V_T and its second through nth elastic degrees of freedom η_i $(i = 2, 3, \ldots, n)$. The resulting short-period approximation for a flexible flight vehicle consists of three linearized first-order differential equations in the three degrees of freedom α, θ, and η_1. The Laplace transform of these equations, expressed in polynomial matrix form, is given by

Table 2 Numerical values for stability derivatives of configurations analyzed

	Numerical values		
Stability derivative	Flexible aircraft[a]	Supersonic missile[b]	Hypersonic aircraft[c]
F_{1_q}	-7.835×10^1	1.110×10^2	-2.374×10^1
F_{1_α}	-1.040×10^3	-4.161×10^2	3.108×10^3
F_{1_δ}	-8.656×10^2	8.920×10^3	0
M_q	-8.302×10^{-1}	-1.410×10^{-1}	-5.294×10^{-2}
M_α	-3.330	-5.943×10^{-1}	4.255
M_δ	-5.115	2.435	-2.322
M_{η_1}	-6.549×10^{-2}	2.567×10^{-3}	1.252×10^{-1}
$M_{\dot\eta_1}$	-3.900×10^{-3}	1.097×10^{-6}	-9.141×10^{-4}
$1 + Z_q/V_T$	1.025	1.198×10^1	1.000
Z_α/V_T	-4.158×10^{-1}	-1.982×10^{-1}	-5.809×10^{-2}
Z_δ/V_T	-8.021×10^{-2}	-2.126×10^{-1}	-1.439×10^{-2}
$1 + Z_{\eta_1}/V_T$	-2.666×10^{-3}	2.519×10^{-4}	-6.825×10^{-4}
$1 + Z_{\dot\eta_1}/V_T$	-1.106×10^{-4}	-6.674×10^{-8}	3.815×10^{-6}
$2\zeta_1\omega_1 - F_{1_{\dot\eta_1}}$	6.214×10^{-1}	6.960×10^{-2}	7.702×10^{-1}
$\omega_1^2 - F_{1_{\eta_1}}$	3.483×10^1	7.365×10^1	2.697×10^2

[a]From Refs. 2, 4. [b]From Ref. 3. [c]From Ref. 6.

$$\begin{bmatrix} s - \dfrac{Z_\alpha}{V_T} & -\left(1 + \dfrac{Z_q}{V_T}\right)s & \dfrac{Z_{\dot\eta_1}}{V_T}s - \dfrac{Z_{\eta_1}}{V_T} \\ -M_\alpha & s^2 - M_q s & -M_{\dot\eta_1}s - M_{\eta_1} \\ -F_{1_\alpha} & -F_{1_q}s & s^2 + \left(2\zeta_1\omega_1 - F_{1_{\dot\eta_1}}\right)s + \left(\omega_1^2 - F_{1_{\eta_1}}\right) \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \theta(s) \\ \eta_1(s) \end{bmatrix} = \begin{bmatrix} \dfrac{Z_\delta}{V_T} \\ M_\delta \\ F_{1_\delta} \end{bmatrix} \delta(s) \tag{1}$$

Equation (1) relates the dynamic responses of the vehicle angle of attack, pitch attitude angle, and generalized coordinate of the first aeroelastic mode to the dynamics of the pitch control surface deflection. Defining the transfer function matrix $G(s)$ of the system by

$$G(s) \equiv \begin{bmatrix} \dfrac{\alpha(s)}{\delta(s)} \\ \dfrac{\theta(s)}{\delta(s)} \\ \dfrac{\eta_1(s)}{\delta(s)} \end{bmatrix} = \begin{bmatrix} s - \dfrac{Z_\alpha}{V_T} & -\left(1 + \dfrac{Z_q}{V_T}\right)s & \dfrac{Z_{\dot\eta_1}}{V_T}s - \dfrac{Z_{\eta_1}}{V_T} \\ -M_\alpha & \left(s^2 - M_q s\right) & \left(-M_{\dot\eta_1}s - M_{\eta_1}\right) \\ -F_{1_\alpha} & -F_{1_q}s & s^2 + \left(2\zeta_1\omega_1 - F_{1_{\dot\eta_1}}\right)s + \left(\omega_1^2 - F_{1_{\eta_1}}\right) \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{Z_\delta}{V_T} \\ M_\delta \\ F_{1_\delta} \end{bmatrix} \tag{2}$$

one can manipulate Eq. (1) to obtain

$$\begin{bmatrix} \alpha(s) \\ \theta(s) \\ \eta_1(s) \end{bmatrix} = G(s)\delta(s) = \begin{bmatrix} \dfrac{\alpha(s)}{\delta(s)} \\ \dfrac{\theta(s)}{\delta(s)} \\ \dfrac{\eta_1(s)}{\delta(s)} \end{bmatrix} \delta(s) \tag{3}$$

The transfer function of any translational/rotational displacement with respect to the elevator input $\delta(s)$ can be expressed as a linear combination of $\alpha(s)/\delta(s)$, $\theta(s)/\delta(s)$, and $\eta_1(s)/\delta(s)$ for any point on the airframe. Literal expressions for the poles and zeros of $\alpha(s)/\delta(s)$, $\theta(s)/\delta(s)$, and $\eta_1(s)/\delta(s)$ are therefore useful for the analysis, design, and simulation of flexible flight vehicles.

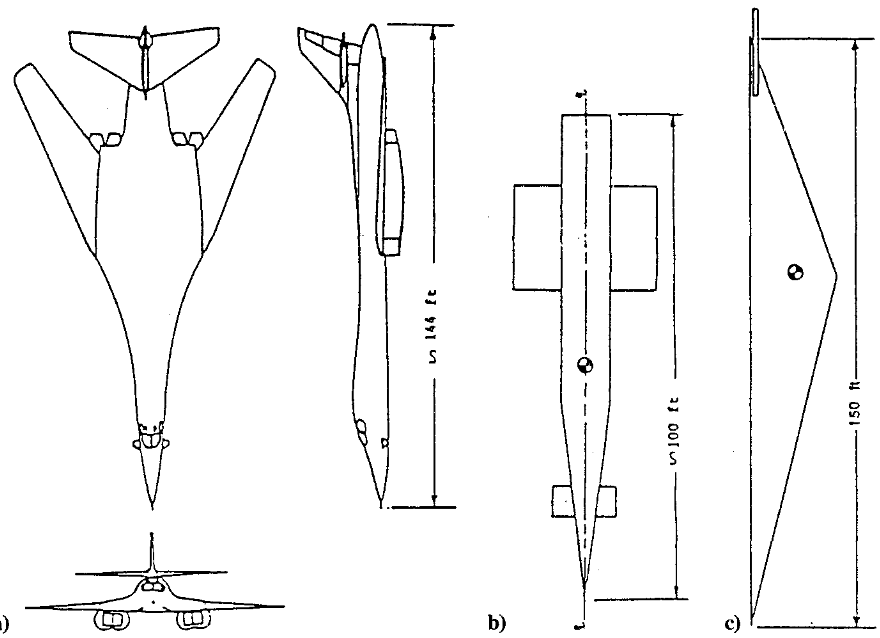

Fig. 1 Configurations analyzed: a) flexible aircraft, b) supersonic missile, and c) hypersonic vehicle.

Automation of Literal Approximation Method

In the literal approximation method we seek to obtain simple analytical expressions for the poles and zeros of the transfer function matrix $G(s)$ of a flexible flight vehicle in terms of the vehicle's stability derivatives. A nominal set of literal approximations based on a nominal polynomial in a factored form is developed. A first-order perturbation about the nominal roots of the nominal polynomial is then analytically derived. Since the resulting literal approximations are quite lengthy, an ordering scheme is used to retain only numerically significant terms, resulting in short literal approximations for the roots of the polynomial analyzed. In this section the automated literal approximation method, resulting from the incorporation of the literal approximation method[2] with the Theorist symbolic manipulation software,[6] is described and exemplified for the flexible aircraft of Fig. 1a.

The elements $\alpha(s)/\delta(s)$, $\theta(s)/\delta(s)$, and $\eta_1(s)/\delta(s)$ of the transfer function $G(s)$ in Eq. (2) are derived using Cramer's rule.[1,2] For example, the numerator $N(s)$ and denominator $D(s)$ of $\alpha(s)/\delta(s)$ are given by

$$N(s) = \det\begin{bmatrix} \dfrac{Z_\delta}{V_T} & -\left(1+\dfrac{Z_q}{V_T}\right)s & -\dfrac{Z_{\dot\eta_1}}{V_T}s - \dfrac{Z_{\eta_1}}{V_T} \\ M_\delta & s^2 - M_q s & -M_{\dot\eta_1}s - M_{\eta_1} \\ F_{1_\delta} & -F_{1_q}s & s^2 + \left(2\zeta_1\omega_1 - F_{1_{\dot\eta 1}}\right)s + \left(\omega_1^2 - F_{1_{\eta 1}}\right) \end{bmatrix} \tag{4}$$

$$D(s) = \det\begin{bmatrix} s - \dfrac{Z_\alpha}{V_T} & -\left(1+\dfrac{Z_q}{V_T}\right)s & -\dfrac{Z_{\dot\eta_1}}{V_T}s - \dfrac{Z_{\eta_1}}{V_T} \\ -M_\alpha & s^2 - M_q s & \left(-M_{\dot\eta_1}s - M_{\eta_1}\right) \\ -F_{1_\alpha} & -F_{1_q}s & s^2 + \left(2\zeta_1\omega_1 - F_{1_{\dot\eta 1}}\right)s + \left(\omega_1^2 - F_{1_{\eta 1}}\right) \end{bmatrix} \tag{5}$$

To obtain literal approximations for the roots of the characteristic polynomial $D(s)$, substitute the flexible aircraft stability parameters of Table 2 into Eq. (5) to get

$$\begin{aligned} D(s) &= s^5 + 1.8674s^4 + 38.942s^3 + 33.479s^2 + 57.355s \\ &= s(s^2 + 0.87442s + 1.5712)(s^2 + 0.99298s + 36.503) \\ &= s^5 + d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s \\ &= s\left(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2\right)\left(s^2 + 2\zeta_{f_1}\omega_{f_1}s + \omega_{f_1}^2\right) \\ &= s(s^2 + Zs + \Omega)\left(s^2 + Z_1 s + \Omega_1\right) \end{aligned} \tag{6}$$

where d_1, d_2, d_3, and d_4 are the coefficients of the characteristic polynomial and the pairs ($2\zeta_{sp}\omega_{sp} = 0.87442$, $\omega_{sp}^2 = 1.5712$) and ($2\zeta_{f_1}\omega_{f_1} = 0.99298$, $\omega_{f_1}^2 = 36.503$), correspond to the exact natural frequency and damping of the short-period mode and the first aeroelastic mode, respectively. The notation $Z \equiv 2\zeta_{sp}\omega_{sp}$, $\Omega \equiv \omega_{sp}^2$, $Z_1 \equiv 2\zeta_{f_1}\omega_{f_1}$, and $\Omega_1 \equiv \omega_{f_1}^2$ is introduced to simplify the ensuing derivation of the literal approximations for the short-period mode of the elastic aircraft of Fig. 1. The characteristic polynomial $D(s)$ of Eq. (6) can be factored into

$$\begin{aligned} D(s) &= s(s + 0.4372 - 1.175i)(s + 0.4372 + 1.175i) \\ &\quad \times (s + 0.4965 - 6.021i)(s + 0.4965 + 6.021i) \end{aligned} \tag{7}$$

The procedure used here to select the nominal polynomial $\bar{D}(s)$, associated with the characteristic polynomial $D(s)$, is to identify all possible fifth-order polynomials in s that can be extracted in a factored literal form from $D(s)$. From the candidate, so identified, one would choose $\bar{D}(s)$ as the polynomial whose roots are closest to the roots of $D(s)$. Examination of the structure of $D(s)$ of Eq. (5) reveals that the only possible candidates for the nominal characteristic polynomial are

$$\begin{aligned} D_1(s) &= s\left[s^2 + \left(-M_q - \frac{Z_\alpha}{V_T}\right)s + M_q\frac{Z_\alpha}{V_T}\right] \\ &\quad \times \left[s^2 + \left(2\zeta_1\omega_1 - F_{1_{\dot\eta 1}}\right)s + \left(\omega_1^2 - F_{1_{\eta 1}}\right)\right] \end{aligned} \tag{8}$$

$$\begin{aligned} D_2(s) &= s\left\{s^2 + \left(-M_q - \frac{Z_\alpha}{V_T}\right)s + \left[M_q\frac{Z_\alpha}{V_T} - M_\alpha\left(1 + \frac{Z_q}{V_T}\right)\right]\right\} \\ &\quad \times \left[s^2 + \left(2\zeta_1\omega_1 - F_{1_{\dot\eta 1}}\right)s + \left(\omega_1^2 - F_{1_{\eta 1}}\right)\right] \end{aligned} \tag{9}$$

Evaluating these polynomials for the flexible aircraft stability derivatives of Table 2 and solving for their roots yield

$$D_1(s) = s(s+0.4158)(s+0.8302)(s+0.3107+5.8935i) \times (s+0.3107-5.8935i) \quad (10)$$

$$D_2(s) = s(s+0.623+1.8358i)(s+0.623-1.8358i) \times (s+0.3107+5.8935i)(s+0.3107-5.8935i) \quad (11)$$

The distance measure l_1 between the roots of $D_1(s)$ and $D(s)$ is given by

$$l_1 = \{(0.4372-0.4158)^2 + 1.175^2 + (0.4372-0.8302)^2 + 1.175^2 + 2[(0.4965-0.3107)^2 + (6.021-5.8935)^2]\}^{\frac{1}{2}} = 1.737 \quad (12)$$

whereas the distance measure l_2 between the roots of $D_2(s)$ and $D(s)$ is given by

$$l_2 = \{2[(0.4372-0.623)^2 + (1.175-1.8358)^2 + (0.4965-0.3107)^2 + (6.021-5.8935)^2]\}^{\frac{1}{2}} = 1.022 \quad (13)$$

Since $l_2 < l_1$, the nominal characteristic polynomial $\bar{D}(s)$ is chosen to be

$$\begin{aligned}\bar{D}(s) &= D_2(s)\\ &= s\left(s^2 + 2\bar{\zeta}_{sp}\bar{\omega}_{sp}s + \bar{\omega}_{sp}^2\right)\left(s^2 + 2\bar{\zeta}_{f1}\bar{\omega}_{f1}s + \bar{\omega}_{f1}^2\right)\\ &= s(s^2 + \bar{Z}s + \bar{\Omega})\left(s^2 + \bar{Z}_1 s + \bar{\Omega}_1\right)\\ &= s^5 + \bar{d}_4 s^4 + \bar{d}_3 s^3 + \bar{d}_2 s^2 + \bar{d}_1 s \end{aligned}\quad (14)$$

where $\bar{d}_1, \bar{d}_2, \bar{d}_3$, and $\bar{d}_4$ are the coefficients of the nominal polynomial and the nominal values $\bar{Z} \equiv 2\bar{\zeta}_{sp}\bar{\omega}_{sp}$, $\bar{\Omega} \equiv \bar{\omega}_{sp}^2$, $\bar{Z}_1 \equiv 2\bar{\zeta}_{f1}\bar{\omega}_{f1}$, and $\bar{\Omega}_1 \equiv \bar{\omega}_{f1}^2$ are given by

$$\bar{\Omega} = M_q\frac{Z_\alpha}{V_T} - M_\alpha\left(1 + \frac{Z_q}{V_T}\right), \qquad \bar{Z} = -M_q - \frac{Z_\alpha}{V_T} \quad (15)$$

for the nominal short-period dynamics and by

$$\bar{\Omega}_1 = \left(\omega_1^2 - F_{1_{\eta 1}}\right), \qquad \bar{Z}_1 = \left(2\zeta_1\omega_1 - F_{1_{\dot{\eta}1}}\right) \quad (16)$$

for the nominal first aeroelastic mode. Note that the nominal pairs $(\bar{\Omega}, \bar{Z})$ and $(\bar{\Omega}_1, \bar{Z}_1)$ of Eqs. (15) and (16) correspond to the traditional natural frequency and damping associated with the rigid-body mode and the first aeroelastic mode of the aircraft, respectively. This, of course, is an end result rather than an initial assumption of the analysis since it is clear that if we had $l_1 < l_2$, we would have had $\bar{D}(s) = D_1(s)$. The literal terms representing the coupling between these two modes are due to the expansion of the exact values of Ω, Z, Ω_1, and Z_1, defined in Eq. (6), about their respective nominal values $\bar{\Omega}$, $\bar{Z}$, $\bar{\Omega}_1$, $\bar{Z}_1$ of Eqs. (14–16). The automated procedure to derive these terms is discussed below.

The main idea underlying the literal approximation method[2] is briefly reviewed here to ensure the clarity of the new results derived below. Using Eqs. (6) and (14), let us define the vectors $\boldsymbol{d}, \bar{\boldsymbol{d}}, \boldsymbol{v}, \bar{\boldsymbol{v}}, \Delta\boldsymbol{d}$, and $\Delta\boldsymbol{v}$ by

$$\boldsymbol{d} \equiv (d_1, d_2, d_3, d_4)^T \quad (17)$$

$$\bar{\boldsymbol{d}} \equiv (\bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4)^T \quad (18)$$

$$\boldsymbol{v} \equiv (\Omega, Z, \Omega_1, Z_1)^T \quad (19)$$

$$\bar{\boldsymbol{v}} \equiv (\bar{\Omega}, \bar{Z}, \bar{\Omega}_1, \bar{Z}_1)^T \quad (20)$$

$$\Delta\boldsymbol{d} \equiv \boldsymbol{d} - \bar{\boldsymbol{d}} \quad (21)$$

$$\Delta\boldsymbol{v} \equiv \boldsymbol{v} - \bar{\boldsymbol{v}} \quad (22)$$

A Taylor series expansion of $\Delta\boldsymbol{d}$ about the origin (i.e., $\boldsymbol{d}$ about $\bar{\boldsymbol{d}}$) yields

$$\Delta\boldsymbol{d}(\Delta\boldsymbol{v}) = \left.\frac{\partial\Delta\boldsymbol{d}(\Delta\boldsymbol{v})}{\partial\Delta\boldsymbol{v}}\right|_{\boldsymbol{v}=\bar{\boldsymbol{v}}}\Delta\boldsymbol{v} + (\text{terms of order} \geq \|\Delta\boldsymbol{v}\|^2) \quad (23)$$

from which

$$\boldsymbol{v} \cong \bar{\boldsymbol{v}} + \left[\left.\frac{\partial\Delta\boldsymbol{d}(\Delta\boldsymbol{v})}{\partial\Delta\boldsymbol{v}}\right|_{\boldsymbol{v}=\bar{\boldsymbol{v}}}\right]^{-1}\Delta\boldsymbol{d} = \bar{\boldsymbol{v}} + \boldsymbol{J}^{-1}\Delta\boldsymbol{d} \quad (24)$$

where

$$\boldsymbol{J} \equiv \left.\frac{\partial\Delta\boldsymbol{d}(\Delta\boldsymbol{v})}{\partial\Delta\boldsymbol{v}}\right|_{\boldsymbol{v}=\bar{\boldsymbol{v}}} \quad (25)$$

Using Eq. (16), one can write Eqs. (5) and (14) as

$$D(s) = \det\begin{bmatrix} s - \frac{Z_\alpha}{V_T} & -\left(1 + \frac{Z_q}{V_T}\right)s & -\frac{Z_{\dot{\eta}1}}{V_T}s - \frac{Z_{\eta 1}}{V_T} \\ -M_\alpha & s^2 - M_q s & -M_{\dot{\eta}1}s - M_{\eta 1} \\ -F_{1_\alpha} & -F_{1_q}s & s^2 + \bar{Z}_1 s + \bar{\Omega}_1 \end{bmatrix} \quad (26)$$

$$\bar{D}(s) = s\left\{s^2 + \left(-M_q - \frac{Z_\alpha}{V_T}\right)s + \left[M_q\frac{Z_\alpha}{V_T} - M_\alpha\left(1 + \frac{Z_q}{V_T}\right)\right]\right\}\left(s^2 + \bar{Z}_1 s + \bar{\Omega}_1\right) \quad (27)$$

Expansion of Eq. (26) for the denominator $D(s)$ yields

$$\begin{aligned} D(s) &= s^5 + d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s\\ &= s^5 + \left(-M_q + \bar{Z}_1 - \frac{Z_\alpha}{V_T}\right)s^4\\ &+ \left[-F_{1_q}M_{\dot{\eta}1} - M_q\bar{Z}_1 - M_\alpha\left(1 + \frac{Z_q}{V_T}\right) + M_q\frac{Z_\alpha}{V_T} - \bar{Z}_1\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\dot{\eta}1}}{V_T} + \bar{\Omega}_1\right]s^3\\ &+ \left[F_{1_\alpha}\frac{Z_{\dot{\eta}1}}{V_T}M_q - F_{1_q}\frac{Z_{\dot{\eta}1}}{V_T}M_\alpha - F_{1_q}M_{\eta 1} - F_{1_\alpha}M_{\dot{\eta}1}\left(1 + \frac{Z_q}{V_T}\right) - M_\alpha\bar{Z}_1\left(1 + \frac{Z_q}{V_T}\right) + F_{1_q}M_{\dot{\eta}1}\frac{Z_\alpha}{V_T} + M_q\bar{Z}_1\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\eta 1}}{V_T} - M_q\bar{\Omega}_1 - \frac{Z_\alpha}{V_T}\bar{\Omega}_1\right]s^2\\ &+ \left[F_{1_\alpha}\frac{Z_{\eta 1}}{V_T}M_q - F_{1_q}\frac{Z_{\eta 1}}{V_T}M_\alpha - F_{1_\alpha}M_{\eta 1}\frac{Z_q}{V_T} + F_{1_q}M_{\eta 1}\frac{Z_\alpha}{V_T} - M_\alpha\left(1 + \frac{Z_q}{V_T}\right)\bar{\Omega}_1 + M_q\frac{Z_\alpha}{V_T}\bar{\Omega}_1\right]s \end{aligned} \quad (28)$$

which has the coefficient vector

$$
\boldsymbol{d} \equiv \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}
= \begin{bmatrix}
F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}M_q - F_{1_q}\frac{Z_{\eta_1}}{V_T}M_\alpha - F_{1_\alpha}M_{\eta_1}\frac{Z_q}{V_T} + F_{1_q}M_{\eta_1}\frac{Z_\alpha}{V_T} - M_\alpha\left(1+\frac{Z_q}{V_T}\right)\bar{\Omega}_1 + M_q\frac{Z_\alpha}{V_T}\bar{\Omega}_1 \\
F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}M_q - F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}M_\alpha - F_{1_q}M_{\eta_1} - F_{1_\alpha}M_{\dot{\eta}_1}\left(1+\frac{Z_q}{V_T}\right) - M_\alpha\bar{Z}_1\left(1+\frac{Z_q}{V_T}\right) + F_{1_q}M_{\dot{\eta}_1}\frac{Z_\alpha}{V_T} + M_q\bar{Z}_1\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T} - M_q\bar{\Omega}_1 - \frac{Z_\alpha}{V_T}\bar{\Omega}_1 \\
-F_{1_q}M_{\dot{\eta}_1} - M_q\bar{Z}_1 - M_\alpha\left(1+\frac{Z_q}{V_T}\right) + M_q\frac{Z_\alpha}{V_T} - \bar{Z}_1\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T} + \bar{\Omega}_1 \\
-M_q + \bar{Z}_1 - \frac{Z_\alpha}{V_T}
\end{bmatrix} \tag{29}
$$

whereas expansion of Eq. (27) for the nominal denominator $\bar{D}(s)$ and the nominal coefficient vector $\bar{\boldsymbol{d}}$ yields

$$
\begin{aligned}
\bar{D}(s) &= s^5 + \bar{d}_4 s^4 + \bar{d}_3 s^3 + \bar{d}_2 s^2 + \bar{d}_1 s \\
&= s^5 + \left(-M_q + \bar{Z}_1 - \frac{Z_\alpha}{V_T}\right)s^4 \\
&\quad + \left[-M_q\bar{Z}_1 - M_\alpha\left(1+\frac{Z_q}{V_T}\right) + M_q\frac{Z_\alpha}{V_T} - \bar{Z}_1\frac{Z_\alpha}{V_T} + \bar{\Omega}_1\right]s^3 \\
&\quad + \left[-M_\alpha\bar{Z}_1\left(1+\frac{Z_q}{V_T}\right) + M_q\bar{Z}_1\frac{Z_\alpha}{V_T} - M_q\bar{\Omega}_1 - \frac{Z_\alpha}{V_T}\bar{\Omega}_1\right]s^2 \\
&\quad + \left(-M_\alpha\frac{Z_q}{V_T}\bar{\Omega}_1 + M_q\frac{Z_\alpha}{V_T}\bar{\Omega}_1\right)s
\end{aligned} \tag{30}
$$

which has the coefficient vector

$$
\bar{\boldsymbol{d}} \equiv \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \bar{d}_4 \end{bmatrix}
= \begin{bmatrix}
-M_\alpha\frac{Z_q}{V_T}\bar{\Omega}_1 + M_q\frac{Z_\alpha}{V_T}\bar{\Omega}_1 \\
-M_\alpha\bar{Z}_1\left(1+\frac{Z_q}{V_T}\right) + M_q\bar{Z}_1\frac{Z_\alpha}{V_T} - M_q\bar{\Omega}_1 - \frac{Z_\alpha}{V_T}\bar{\Omega}_1 \\
-M_q\bar{Z}_1 - M_\alpha\left(1+\frac{Z_q}{V_T}\right) + M_q\frac{Z_\alpha}{V_T} - \bar{Z}_1\frac{Z_\alpha}{V_T} + \bar{\Omega}_1 \\
-M_q + \bar{Z}_1 - \frac{Z_\alpha}{V_T}
\end{bmatrix} \tag{31}
$$

Use of Eqs. (29) and (31) to calculate the coefficient vector $\boldsymbol{\Delta d} \equiv \boldsymbol{d} - \bar{\boldsymbol{d}}$ of the polynomial $\Delta D(s) = D(s) - \bar{D}(s)$ yields

$$
\boldsymbol{\Delta d} = \begin{bmatrix}
F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}M_q - F_{1_q}\frac{Z_{\eta_1}}{V_T}M_\alpha - F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right) + F_{1_q}M_{\eta_1}\frac{Z_\alpha}{V_T} \\
F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}M_q - F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}M_\alpha - F_{1_q}M_{\eta_1} - F_{1_\alpha}M_{\dot{\eta}_1}\left(1+\frac{Z_q}{V_T}\right) + F_{1_q}M_{\dot{\eta}_1}\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T} \\
-F_{1_q}M_{\dot{\eta}_1} - F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T} \\
0
\end{bmatrix} \tag{32}
$$

By expanding the last row of Eq. (6) for $D(s)$, one can evaluate the coefficient vector $\boldsymbol{d}$ and the Jacobian $\boldsymbol{J}$ to obtain

$$
\boldsymbol{d} = \begin{bmatrix} \Omega\Omega_1 \\ Z\Omega_1 + Z_1\Omega \\ ZZ_1 + \Omega_1 + \Omega \\ Z_1 + Z \end{bmatrix}
$$

$$
\boldsymbol{J} = \left[\frac{\partial \boldsymbol{d}(\boldsymbol{v})}{\partial \boldsymbol{v}}\right]\Bigg|_{\boldsymbol{v}=\bar{\boldsymbol{v}}} = \begin{bmatrix} \bar{\Omega}_1 & 0 & \bar{\Omega} & 0 \\ \bar{Z}_1 & \bar{\Omega}_1 & \bar{Z} & \bar{\Omega} \\ 1 & \bar{Z}_1 & 1 & \bar{Z} \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{33}
$$

Inversion of $\boldsymbol{J}$, of Eq. (33), yields

$$
\boldsymbol{J}^{-1} = \frac{1}{|\boldsymbol{J}|} \times \begin{bmatrix}
D_\Omega - D_Z\bar{Z} & D_Z\bar{\Omega} & -D_\Omega\bar{\Omega} & D_{\Omega Z}\bar{\Omega} \\
-D_Z & D_\Omega & -D_{\Omega Z} & D_{\Omega Z}\bar{Z} - D_\Omega\bar{\Omega} \\
-D_\Omega + D_Z\bar{Z}_1 & -D_Z\bar{\Omega}_1 & D_\Omega\bar{\Omega}_1 & -D_{\Omega Z}\bar{\Omega}_1 \\
D_Z & -D_\Omega & D_{\Omega Z} & -D_{\Omega Z}\bar{Z}_1 + D_\Omega\bar{\Omega}_1
\end{bmatrix} \tag{34}
$$

where

$$
D_Z \equiv \bar{Z}_1 - \bar{Z}, \qquad D_\Omega \equiv \bar{\Omega}_1 - \bar{\Omega}
$$
$$
D_{\Omega Z} \equiv \bar{Z}\bar{\Omega}_1 - \bar{Z}_1\bar{\Omega}, \qquad |\boldsymbol{J}| = D_\Omega^2 - D_Z D_{\Omega Z} \tag{35}
$$

Substitution of Eqs. (32) and (34) into Eq. (24) results in the exact literal approximationfor $\boldsymbol{\Delta v} \equiv \boldsymbol{v} - \bar{\boldsymbol{v}} \equiv (\Delta\Omega, \Delta Z, \Delta\Omega_1,$

$\Delta Z_1)^T$, given by

$$\mathbf{\Delta v} = \frac{1}{D_\Omega^2 - D_Z D_{\Omega Z}} \times \begin{bmatrix} \begin{gathered} +F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_\Omega M_q - F_{1_q}\frac{Z_{\eta_1}}{V_T}D_\Omega M_\alpha - F_{1_\alpha}M_{\eta_1}D_\Omega\left(1+\frac{Z_q}{V_T}\right) + F_{1_\alpha}M_{\eta_1}D_Z\bar{Z}\left(1+\frac{Z_q}{V_T}\right) + F_{1_q}M_{\eta_1}D_\Omega\frac{Z_\alpha}{V_T} \\ -F_{1_q}M_{\eta_1}D_\Omega - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_\Omega + F_{1_q}M_{\dot{\eta}_1}D_{\Omega Z} + F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_{\Omega Z} \end{gathered} \\ \begin{gathered} -F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_\Omega M_q + F_{1_q}\frac{Z_{\eta_1}}{V_T}D_\Omega M_\alpha + F_{1_\alpha}M_{\eta_1}D_\Omega\left(1+\frac{Z_q}{V_T}\right) - F_{1_\alpha}M_{\eta_1}D_Z\bar{Z}_1\left(1+\frac{Z_q}{V_T}\right) - F_{1_q}M_{\eta_1}D_\Omega\frac{Z_\alpha}{V_T} \\ +F_{1_q}M_{\eta_1}D_\Omega + F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_\Omega - F_{1_q}M_{\dot{\eta}_1}D_{\Omega Z} - F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_{\Omega Z} \end{gathered} \\ \begin{gathered} -F_{1_q}M_{\eta_1}D_Z\bar{Z}\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z M_q\bar{Z} + F_{1_q}\frac{Z_{\eta_1}}{V_T}D_Z M_\alpha\bar{Z} - F_{1_q}M_{\eta_1}D_Z\bar{\Omega} - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z\bar{\Omega} \\ -F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z M_q + F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega M_q + F_{1_q}\frac{Z_{\eta_1}}{V_T}D_Z M_\alpha - F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega M_\alpha \\ +F_{1_q}M_{\eta_1}D_Z\bar{Z}_1\frac{Z_\alpha}{V_T} + F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z M_q\bar{Z}_1 - F_{1_q}\frac{Z_{\eta_1}}{V_T}D_Z M_\alpha\bar{Z}_1 + F_{1_q}M_{\eta_1}D_Z\bar{\Omega}_1 + F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z\bar{\Omega}_1 \\ +F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z M_q - F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega M_q - F_{1_q}\frac{Z_{\eta_1}}{V_T}D_Z M_\alpha + F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega M_\alpha \end{gathered} \\ \begin{gathered} +F_{1_q}M_{\dot{\eta}_1}D_\Omega\bar{\Omega} + F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega\bar{\Omega} + F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_Z M_q\bar{\Omega} - F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}D_Z M_\alpha\bar{\Omega} - F_{1_\alpha}M_{\dot{\eta}_1}D_Z\left(1+\frac{Z_q}{V_T}\right)\bar{\Omega} + F_{1_q}M_{\dot{\eta}_1}D_Z\frac{Z_\alpha}{V_T}\bar{\Omega} \\ +F_{1_\alpha}M_{\eta_1}D_Z\left(1+\frac{Z_q}{V_T}\right) - F_{1_\alpha}M_{\dot{\eta}_1}D_\Omega\left(1+\frac{Z_q}{V_T}\right) - F_{1_q}M_{\eta_1}D_Z\frac{Z_\alpha}{V_T} + F_{1_q}M_{\dot{\eta}_1}D_\Omega\frac{Z_\alpha}{V_T} \\ -F_{1_q}M_{\dot{\eta}_1}D_\Omega\bar{\Omega}_1 - F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega\bar{\Omega}_1 - F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_Z M_q\bar{\Omega}_1 + F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}D_Z M_\alpha\bar{\Omega}_1 + F_{1_\alpha}M_{\dot{\eta}_1}D_Z\left(1+\frac{Z_q}{V_T}\right)\bar{\Omega}_1 - F_{1_q}M_{\dot{\eta}_1}D_Z\frac{Z_\alpha}{V_T}\bar{\Omega}_1 \\ -F_{1_\alpha}M_{\eta_1}D_Z\left(1+\frac{Z_q}{V_T}\right) + F_{1_\alpha}M_{\dot{\eta}_1}D_\Omega\left(1+\frac{Z_q}{V_T}\right) + F_{1_q}M_{\eta_1}D_Z\frac{Z_\alpha}{V_T} - F_{1_q}M_{\dot{\eta}_1}D_\Omega\frac{Z_\alpha}{V_T} \end{gathered} \end{bmatrix} \tag{36}$$

This literal approximation is lengthy and as a result impractical for design and simulation purposes. Shorter literal expressions for $\mathbf{\Delta v}$ can be obtained, however, by retaining only the numerically significant terms in the numerators and the denominators of the four entries of $\mathbf{\Delta v}$. To do this, evaluate the numerical value of each of the 56 literal terms appearing in the four entries of $\mathbf{\Delta v}$ on the right-hand side of Eq. (36) and retain in each entry only those literal terms with absolute values that exceed the corresponding entry of the limit vector $\mathbf{v}_L$, given by

$$\mathbf{v}_L \equiv p|\mathbf{J}|\mathbf{v} = p|\mathbf{J}|\begin{bmatrix}\Omega \\ Z \\ \Omega_1 \\ Z_1\end{bmatrix} = 0.1 \times 991 \times \begin{bmatrix}1.57 \\ 0.87 \\ 36.5 \\ 0.99\end{bmatrix} = \begin{bmatrix}155.7 \\ 86.7 \\ 3617 \\ 98.4\end{bmatrix} \tag{37}$$

The variables Z, Ω, Z_1, Ω_1, and $|\mathbf{J}|$ in Eq. (37) are evaluated for the flexible aircraft based on Eqs. (6), (33), (35), (15), and (16) and Table 2, whereas the scalar p in Eq. (37) corresponds to the magnitude of the error allowed in the approximation process. For example, the current choice of $p = 0.1$ means that terms that contribute less than 10% of the true value of the product ($|\mathbf{J}|\mathbf{v}$) are discarded. A decrease in the value of p (e.g., $p = 0.05$) would result in more terms being retained in the literal approximation leading to better accuracy at the price of more complicated literal expressions. The same methodology is also applied to $|\mathbf{J}|$, the common denominator of all the elements of $\mathbf{\Delta v}$. The limit $|\mathbf{J}|_L$ on the elements to be retained in $|\mathbf{J}|$ is given by

$$|\mathbf{J}|_L \equiv p|\mathbf{J}| = 0.1 \times 991.09 = 99.109 \tag{38}$$

To demonstrate the approximation process, let us evaluate the literal approximation for the square of the short-period frequency Ω ($= \bar{\Omega} + \Delta\Omega$), which is the first element of $\mathbf{v}$. The exact expression for the numerator of $\Delta\Omega$ is

$$\begin{aligned} |\mathbf{J}|\Delta\Omega = {} & F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_\Omega M_q - F_{1_q}\frac{Z_{\eta_1}}{V_T}D_\Omega M_\alpha \\ & - F_{1_\alpha}M_{\eta_1}D_\Omega\left(1+\frac{Z_q}{V_T}\right) + F_{1_\alpha}M_{\eta_1}D_Z\bar{Z}\left(1+\frac{Z_q}{V_T}\right) \\ & + F_{1_q}M_{\eta_1}D_\Omega\frac{Z_\alpha}{V_T} - F_{1_q}M_{\eta_1}D_Z\bar{Z}\frac{Z_\alpha}{V_T} - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z M_q\bar{Z} \\ & + F_{1_q}\frac{Z_{\eta_1}}{V_T}D_Z M_\alpha\bar{Z} - F_{1_q}M_{\eta_1}D_Z\bar{\Omega} - F_{1_\alpha}\frac{Z_{\eta_1}}{V_T}D_Z\bar{\Omega} \\ & + F_{1_q}M_{\dot{\eta}_1}D_\Omega\bar{\Omega} + F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_\Omega\bar{\Omega} + F_{1_\alpha}\frac{Z_{\dot{\eta}_1}}{V_T}D_Z M_q\bar{\Omega} \\ & - F_{1_q}\frac{Z_{\dot{\eta}_1}}{V_T}D_Z M_\alpha\bar{\Omega} - F_{1_\alpha}M_{\dot{\eta}_1}D_Z\left(1+\frac{Z_q}{V_T}\right)\bar{\Omega} \\ & + F_{1_q}M_{\dot{\eta}_1}D_Z\frac{Z_\alpha}{V_T}\bar{\Omega} \end{aligned} \tag{39}$$

Evaluation of Eq. (39) using the stability derivatives of the flexible aircraft given in Table 2 yields

$$\begin{aligned} -2264.7 = {} & -71.522 + 21.613 - 2169.2 - 54.332 - 66.292 \\ & - 1.6604 - 1.7914 + 0.54133 + 12.045 + 6.5088 + 35.684 \\ & + 13.433 + 0.22417 - 0.067741 + 9.7596 + 0.29826 \end{aligned} \tag{40}$$

Clearly the third term, $-F_{1_\alpha}M_{\eta_1}D_\Omega(1+Z_q/V_T) = -2169.2$, on the right-hand sides of Eqs. (39) and (40) is the only term with absolute value exceeding the value of $v_L(1) = 155.72$ defined in Eq. (37).

Retaining only this term, the numerator of the first entry of $\Delta\boldsymbol{v}$ is approximated by

$$|\boldsymbol{J}|\Delta\Omega \cong -F_{1_\alpha}M_{\eta_1}D_\Omega\left(1+\frac{Z_q}{V_T}\right) \tag{41}$$

The exact denominator $|\boldsymbol{J}|$ of the first entry of $\Delta\boldsymbol{v}$ is given by

$$|\boldsymbol{J}| = D_\Omega^2 - D_Z D_{\Omega Z} \tag{42}$$

Evaluation of Eq. (42) using the stability derivatives of the flexible aircraft given in Table 2 yields $991.09 = 965.44 + 25.648$. Clearly $D_\Omega^2 = 965.44$ is the only term whose absolute value exceeds the value of $|\boldsymbol{J}|_L = 99.109$ defined in Eq. (38). Retaining only this term, the denominator of the first entry of $\Delta\boldsymbol{v}$ is approximated by dividing Eq. (41) by $|\boldsymbol{J}| \cong D_\Omega^2$. The result is

$$\Delta\Omega \cong -\frac{1}{D_\Omega}F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right) \tag{43}$$

Substitution of Eq. (43) into Eq. (24) yields

$$\Omega \cong \bar{\Omega} - \frac{1}{D_\Omega}F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right) \tag{44}$$

Recalling that $\Omega \equiv \omega_{sp}^2$ and using Eqs. (35), (15), and (16) for $D\Omega$, $\bar{\Omega}$, and $\bar{\Omega}_1$ yield

$$\omega_{sp}^2 \cong \boxed{-M_\alpha\left(1+\frac{Z_q}{V_T}\right)+M_q\frac{Z_\alpha}{V_T}}$$
$$-\left[\omega_1^2 - F_{1_{\eta_1}} + M_\alpha\left(1+\frac{Z_q}{V_T}\right) - M_q\frac{Z_\alpha}{V_T}\right]^{-1}$$
$$\times F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right) = 1.5116\ \text{rad/s}^2 \tag{45}$$

The terms enclosed in the box correspond to the nominal value of the short-period approximation frequency $\bar{\Omega} \equiv \bar{\omega}_{sp}^2$. Equation (45) is identical to Eq. (A5) in Appendix A. The error in the approximation (45) is determined by comparing its value $\omega_{sp}^2 \cong 1.5116$ to the exact value of $\omega_{sp}^2 = 1.5712$ given in Eq. (6). In this case the percent error is given by $100(1.5116/1.5712 - 1) = -3.8\%$.

Unified Set of Literal Approximations

Given a general flexible flight vehicle model in a polynomial matrix form together with the numerical values of the stability derivatives for a specific configuration, one can use the literal approximation method to obtain literal approximations for the poles and zeros of $\alpha(s)/\delta(s)$, $\theta(s)/\delta(s)$, and $\eta_1(s)/\delta(s)$ of this specific vehicle. No guarantee exists, however, that such a set will be identical or even similar to a set of literal approximations derived for a significantly different vehicle. Differences might occur due to different nominal polynomials and/or to retention of different dominant literal terms in the approximation process. When different configurations have identical nominal polynomials, one can use the union of their literal approximations to obtain a unified literal set. We next construct and examine a unified set of literal approximations for three significantly different flexible flight vehicles. The simplicity and generality of such a set is of major significance for assessment of the usefulness of the literal approximation method.

The unified literal approximations for the parameters ω_{sp}^2, $2\zeta_{sp}\omega_{sp}$, $\omega_{f_1}^2$, and $2\zeta_{f_1}\omega_{f_1}$, corresponding to the characteristic polynomial $D(s)$ of Eq. (6), are given by Eqs. (A5–A8) in Appendix A. The terms enclosed in the boxes on the right hand side of each equation in Appendix A correspond to the nominal value of the term on the left-hand side of the same equation. As noted earlier, the literal approximations for $(\bar{\omega}_{sp}^2, 2\bar{\zeta}_{sp}\bar{\omega}_{sp})$ and $(\bar{\omega}_{f_1}^2, 2\bar{\zeta}_{f_1}\bar{\omega}_{f_1})$ correspond to the natural frequency and damping of the rigid-body short-period mode and the uncoupled first aeroelastic mode of the aircraft, respectively. The third term on the right-hand side of each of the approximations (A5–A8) represents the coupling effect between these modes. The expressions reported by Newman and Schmidt[2] are reproduced together with the additional coupling term

$$\frac{2\zeta_1\omega_1 - F_{1_{\dot{\eta}_1}} + M_q + Z_\alpha/V_T}{\left[\omega_1^2 - F_{1_{\eta_1}} + M_\alpha(1+Z_q/V_T) - M_q(Z_\alpha/V_T)\right]^2}F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right)$$

appearing in the approximations for both $2\zeta_{sp}\omega_{sp}$ and $2\zeta_{f_1}\omega_{f_1}$. Also, the denominator of the coupling terms in approximations (A5–A8) was modified from $\Omega_1 \equiv \omega_1^2 - F_{1_{\eta_1}}$ to

$$\Delta\Omega \equiv \Omega_1 - \Omega = \left[\omega_1^2 - F_{1_{\eta_1}} + M_\alpha\left(1+\frac{Z_q}{V_T}\right) - M_q\frac{Z_\alpha}{V_T}\right]$$

where the frequency spacing $\Delta\Omega$ is the difference between the square of the first elastic frequency and the square of short-period mode frequency. The tables in Appendix B demonstrate that some of the rigid-aircraft approximation yield errors with magnitudes larger than 130% for the flexible aircraft, 24% for the hypersonic vehicle, and 2.5% for the missile. When coupling effects are included in the literal approximations, the resulting errors are smaller than 8% for the flexible aircraft, 0.3% for the hypersonic vehicle, and 0.1% for the missile. An interesting observation regarding both the current and the previously derived results[2] is that the nominal frequency and damping of the rigid-body mode are increased/decreased by given amounts whereas their counterparts for the first aeroelastic mode are both decreased/increased by the same amounts. In other words, the rigid-body aeroelastic coupling has approximately equal and opposite effects on the natural frequency and damping associated with the rigid body and the aeroelastic modes. Another observation involves the frequency spacing $\Delta\Omega$. Since $\Delta\Omega$ appears in the denominator of the coupling term in the literal expressions (A6), and (A8) for $2\zeta_{sp}\omega_{sp}$ and $2\zeta_{f_1}\omega_{f_1}$, it is clear that the coupling between the short-period mode and the first elastic mode will increase when their frequencies become closely spaced. If, for example, the first elastic mode frequency of the large flexible aircraft is reduced from 5.902 to 1.968 rad/s due to aerodynamic heating and/or some structural damage, the short-period mode will become unstable. An airframe with closely spaced rigid-body/elastic frequencies will therefore require a stabilizing control loop.

The unified literal approximations for the parameters $1/T_\alpha$, $2\zeta_\alpha\omega_\alpha$, and ω_α^2 appearing on the numerator of the $\alpha(s)/\delta(s)$ transfer function are given by Eqs. (A9–A11) in Appendix A. The literal approximation $1/\bar{T}_\alpha$ of Eq. (A9) for the nominal real zero of $\alpha(s)/\delta(s)$ corresponds to the classical rigid-body approximation. For the case at hand we also have $1/T_\alpha = 1/\bar{T}_\alpha$, which means that the aeroelastic mode has a negligible effect on the real zero of $\alpha(s)/\delta(s)$. The third term on the right-hand side of Eqs. (A10) and (A11) corresponds to the effect of the coupling between the rigid-body and the first aeroelastic mode on the zeros of $\alpha(s)/\delta(s)$, associated with $s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2$. As can be seen from the tables in Appendix B, these coupling terms significantly improve the accuracy of the literal approximations for $2\zeta_\alpha\omega_\alpha$ and ω_α^2. The errors are smaller than 0.2% for the hypersonic vehicle, 0.4% for the missile, and 4% for the flexible aircraft, whereas some of the nominal (rigid-body) approximations exceed 0.2% for the hypersonic vehicle, 68% for the missile, and 300% for the flexible aircraft. Examination of the frequency response plots in Fig. C1 of Appendix C reveals that the unified literal approximations closely match the exact transfer functions associated with the short-period approximation for each of the elastic vehicles. Significant improvements vs the nominal approximation are evident in the phase plots of both the aircraft and the missile.

The unified literal approximations for the parameters $1/T_\theta$, $2\zeta_\theta\omega_\theta$, and ω_θ^2 appearing on the numerator of the $\theta(s)/\delta(s)$ transfer function are given by Eqs. (A12–A14) in Appendix A. The term $1/\bar{T}_\theta$ [enclosed in the box in Eq. (A12)] is the real zero corresponding to the rigid-body approximation.[1] Examination of Appendix A reveals that $\bar{\omega}_\theta = \bar{\omega}_\alpha = \bar{\omega}_{f_1}$ and $\bar{\zeta}_\theta = \bar{\zeta}_\alpha = \bar{\zeta}_{f_1}$, which means that $2\zeta_\theta\omega_\theta$, $2\zeta_\alpha\omega_\alpha$, and $2\zeta_{f_1}\omega_{f_1}$ are all expanded about $2\bar{\zeta}_{f_1}\bar{\omega}_{f_1}$, whereas ω_θ^2, ω_α^2, and $\omega_{f_1}^2$ are all expanded about $\bar{\omega}_{f_1}^2$. In comparison to $\alpha(s)/\delta(s)$, where the coupling between the rigid-body and the aeroelastic mode did not affect the real zero, the rigid-body aeroelastic coupling affects all three zeros of $\theta(s)/\delta(s)$. The numerical values and the errors in these approximations are summarized in Appendix B. These errors are less than 0.1% for the hypersonic vehicle, less than 8% for the missile, and less than 0.4% for the flexible aircraft with the exception of $2\zeta_\theta\omega_\theta$ which is only −435.1% accurate due to the small exact value of $2\zeta_\theta\omega_\theta = -0.0062$. Examination of the frequency response plots of $q(s)/\delta(s)$ [$=s\theta(s)/\delta(s)$] in Fig. C2 of Appendix C reveals that the unified literal approximations closely match the exact transfer functions associated with the short period approximation for an elastic vehicle. The improvements of the literal approximations vs their nominal values are smaller than those encountered for $\alpha(s)/\delta(s)$, especially in the phase plots. The most improved configuration remains the elastic airplane.

Due to the simple algebraic form of the zero polynomial of $\eta_1(s)/\delta(s)$ of Eq. (A3) in Appendix A, one can obtain the exact literal expressions for the parameters $2\zeta_{f_1}\omega_{f_1}$ and $\omega_{f_1}^2$ appearing in its numerator without using the literal approximation method to expand them about their nominal values. The literal approximation for $2\zeta_{f_1}\omega_{f_1}$ and $\omega_{f_1}^2$ given by Eqs. (A15) and (A16) of Appendix A are obtained by neglecting numerically insignificant terms from the exact values of these parameters. The numerical values and the accuracy of these approximations are summarized in Appendix B. The errors are less than 0.1% for the hypersonic vehicle and the missile and less than 1% for the flexible aircraft. These small errors are reflected in the Bode plot of $\eta_1(s)/\delta(s)$, given by Fig. C3 in Appendix C.

Comparison of the accuracy of the nominal vs the unified and the exact parameters associated with the characteristic polynomial $D(s)$ and the numerators of the transfer functions $\alpha(s)/\delta(s)$ and $\theta(s)/\delta(s)$ reveals that the errors obtained by using the nominal (rigid-body) approximations are one to two orders of magnitude larger than the errors obtained using the unified approximations. An improved accuracy of the unified vs the nominal approximation is also reflected in their frequency response plots, as is demonstrated in Appendix C.

Conclusions

A key concern regarding the literal approximation method is its usefulness as an additional analysis/synthesis tool to existing design and simulation procedures. The utility of the method was demonstrated by using symbolic manipulation software to independently derive literal approximations for the longitudinal dynamics of three significantly different flexible flight vehicles, namely an aircraft, a missile, and a hypersonic vehicle. The union of these literal approximations was then formed to get a set of unified literal approximations for the longitudinal dynamics of a generic flexible flight vehicle. This unified set consists of relatively simple literal approximations for the poles and zeros of the transfer functions relating angle of attack, pitch angle, and first-elastic-mode deflection outputs to the elevator deflection input. The effects of the coupling between the rigid-body and the first aeroelastic modes on the poles and zeros of these transfer functions were examined by comparing the numerical values and the frequency response plots resulting from the unified approximations to their traditional decoupled nominal counterparts for the aircraft, missile, and hypersonic vehicle configurations. The unified literal approximations exhibited significant improvement in accuracy over the traditional decoupled approximations, yielding errors smaller than 10% in most cases and providing new and useful insights about the flexibility effects on the longitudinal poles and zeros of a generic flight vehicle.

Appendix A: Unified Literal Approximations for the Factored Transfer Functions

The terms enclosed in boxes on the right-hand side of each equation below correspond to the nominal value of the term on the left-hand side of the same equation. The unified transfer functions associated with the short-period approximation for an elastic vehicle are given by

$$\frac{\alpha(s)}{\delta(s)} = \frac{Z_\delta s(s+1/T_\alpha)\left(s^2+2\zeta_\alpha\omega_\alpha s+\omega_\alpha^2\right)}{D(s)} \tag{A1}$$

$$\frac{\theta(s)}{\delta(s)} = \frac{M_\delta(s+1/T_\theta)\left(s^2+2\zeta_\theta\omega_\theta s+\omega_\theta^2\right)}{D(s)} \tag{A2}$$

$$\frac{\eta_1(s)}{\delta(s)} = \frac{F_{1_\delta}5\left(s^2+2\zeta_{\eta_1}\omega_{\eta_1}s+\omega_{\eta_1}^2\right)}{D(s)} \tag{A3}$$

where

$$D(s) = s\left(s^2+2\zeta_{sp}\omega_{sp}s+\omega_{sp}^2\right)\left(s^2+2\zeta_{f_1}\omega_{f_1}s+\omega_{f_1}^2\right) \tag{A4}$$

$$\omega_{sp}^2 \cong \boxed{-M_\alpha\left(1+\frac{Z_q}{V_t}\right)+M_q\frac{Z_\alpha}{V_T}}$$
$$-\left[\omega_1^2-F_{1_{\eta 1}}+M_\alpha\left(1+\frac{Z_q}{V_T}\right)-M_q\frac{Z_\alpha}{V_T}\right]^{-1}F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right) \tag{A5}$$

$$2\zeta_{sp}\omega_{sp} \cong \boxed{-M_q-\frac{Z_q}{V_T}}$$
$$-\left[\omega_1^2-F_{1_{\eta 1}}+M_\alpha\left(1+\frac{Z_q}{V_T}\right)-M_q\frac{Z_\alpha}{V_T}\right]^{-1}$$
$$\times\left\{F_{1_q}M_{\eta_1}+F_{1_\alpha}\left[\frac{Z_{\eta 1}}{V_T}+\left(1+\frac{Z_q}{V_T}\right)\right.\right.$$
$$\left.\left.\times\left(M_{\dot\eta_1}-\frac{2\zeta_1\omega_1-F_{1_{\dot\eta 1}}+M_q+Z_\alpha/V_T}{\omega_1^2-F_{1_{\eta 1}}+M_\alpha(1+Z_q/V_T)-M_q(Z_\alpha/V_T)}M_{\eta_1}\right)\right]\right\} \tag{A6}$$

$$\omega_{f_1}^2 \cong \boxed{\omega_1^2-F_{1_{\eta 1}}}$$
$$+\left[\omega_1^2-F_{1_{\eta 1}}+M_\alpha\left(1+\frac{Z_q}{V_T}\right)-M_q\frac{Z_\alpha}{V_T}\right]^{-1}$$
$$\times F_{1_\alpha}M_{\eta_1}\left(1+\frac{Z_q}{V_T}\right) \tag{A7}$$

$$2\zeta_{f_1}\omega_{f_1} \cong \boxed{2\zeta_1\omega_1-F_{1_{\eta 1}}}$$
$$+\left[\omega_1^2-F_{1_{\eta 1}}+M_\alpha\left(1+\frac{Z_q}{V_T}\right)-M_q\frac{Z_\alpha}{V_T}\right]^{-1}$$
$$\times\left\{F_{1_q}M_{\eta_1}+F_{1_\alpha}\left[\frac{Z_{\eta 1}}{V_T}+\left(1+\frac{Z_q}{V_T}\right)\right.\right.$$
$$\left.\left.\times\left(M_{\dot\eta_1}-\frac{2\zeta_1\omega_1-F_{1_{\dot\eta 1}}+M_q+Z_\alpha/V_T}{\omega_1^2-F_{1_{\eta 1}}+M_\alpha(1+Z_q/V_T)-M_q(Z_\alpha/V_T)}M_{\eta_1}\right)\right]\right\} \tag{A8}$$

$$\frac{1}{T_\alpha} = \boxed{-M_q+\frac{V_T M_\delta}{Z_\delta}\left(1+\frac{Z_q}{V_T}\right)} \tag{A9}$$

$$2\zeta_\alpha\omega_\alpha = \boxed{2\zeta_1\omega_1-F_{1_{\dot\eta 1}}}+\frac{F_{1_\delta}}{Z_\delta[-M_q+(Z_\delta/V_T)M_\delta(1+Z_q/V_T)]}$$
$$\times\left[\left(1+\frac{Z_q}{V_T}\right)\left(M_{\dot\eta_1}-\frac{M_{\eta_1}}{-M_q+(Z_\delta/V_T)M_\delta(1+Z_q/V_T)}\right)\right.$$
$$\left.+M_{\dot\eta_1}\frac{Z_{\eta_1}}{V_T}\right] \tag{A10}$$

$$\omega_\alpha^2 = \boxed{\omega_1^2 - F_{1_{\eta 1}}} + \frac{F_{1_\delta} M_{\eta 1}}{Z_\delta[-M_q + (Z_\delta/V_T)M_\delta(1+Z_q/V_T)]} \times \left(1+\frac{Z_q}{V_T}\right) \tag{A11}$$

$$\frac{1}{T_\theta} \cong \frac{Z_\delta}{V_T}\left(\frac{M_\alpha}{M_\delta}-\frac{Z_\alpha}{Z_\delta}\right)\left[\boxed{1} + \frac{F_{1_\delta}}{M_\delta(\omega_1^2 - F_{1_{\eta 1}})}\right] \times \left[\frac{M_{\dot\eta 1}}{V_T}\left(\frac{M_\alpha}{M_\delta}-\frac{Z_\alpha}{Z_\delta}\right)+\frac{1}{Z_\delta}(M_\delta Z_{\eta_1} - M_{\eta_1}Z_\delta)\right] \tag{A12}$$

$$2\zeta_\theta\omega_\theta \cong \boxed{2\zeta_1\omega_1 - F_{1_{\dot\eta 1}}} + \frac{F_{1_\delta}M_{\dot\eta 1}}{M_\delta} - \frac{F_{1_\delta}}{M_\delta V_T(\omega_1^2 - F_{1_{\eta 1}})}(M_\delta Z_{\eta_1} - M_{\eta_1}Z_\delta)\left(\frac{M_\alpha}{M_\delta}-\frac{Z_\alpha}{Z_\delta}\right) \tag{A13}$$

$$\omega_\theta^2 \cong \boxed{\omega_1^2 - F_{1_{\eta 1}}} + F_{1_\delta}\frac{M_{\eta 1}}{M_\delta} \tag{A14}$$

$$2\zeta_{\eta_1}\omega_{\eta_1} = -M_q - \frac{Z_\alpha}{V_T} + \frac{1}{F_{1_\delta}}\left(F_{1_q}M_\delta + F_{1_\alpha}\frac{Z_\delta}{V_T}\right) \tag{A15}$$

$$\omega_{\eta_1}^2 \cong -M_\alpha\left(1+\frac{Z_q}{V_T}\right) + M_q\frac{Z_q}{V_T} + \frac{F_{1_\alpha}}{F_{1_\delta}}M_\delta\left(1+\frac{Z_q}{V_T}\right) \tag{A16}$$

Appendix C: Frequency Response Plots of Exact, Approximated, and Nominal $\alpha(s)/\delta(s)$, $\theta(s)/\delta(s)$, and $\eta_1(s)/\delta(s)$

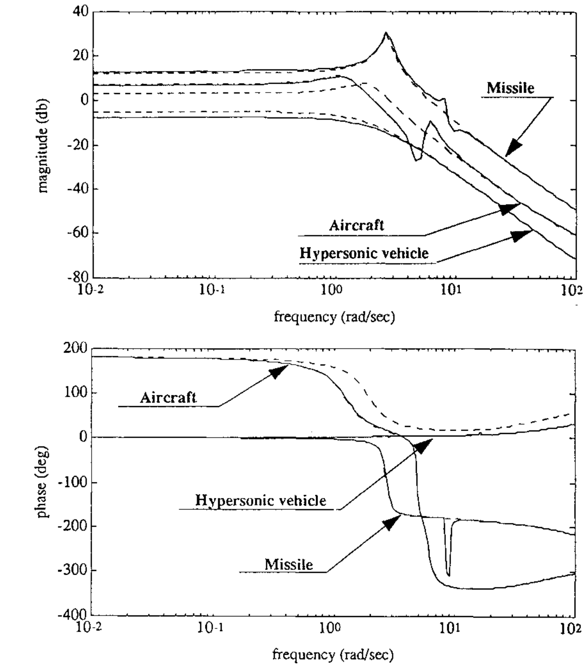

Fig. C1 **$\alpha(s)/\delta(s)$ frequency response for approximate literal models; ——, exact elastic short period model; - - - - -, nominal literal approximation; · · · · · · -, unified literal approximation.**

Appendix B: Values and Accuracy of Literal Approximations of Appendix A

Table B1 Values and accuracy: literal approximations

	Flexible aircraft		Supersonic missile		Hypersonic vehicle	
Parameter	Value	Error (%)	Value	Error (%)	Value	Error (%)
ω_{sp}^2	1.5116	−3.8	7.3406	0.0	−5.6731	0.1
$2\zeta_{sp}\omega_{sp}$	0.81266	−7.1	0.33735	0.0	0.14341	0.2
ω_{f1}^2	37.077	1.6	73.458	0.0	271.17	0.0
$2\zeta_{f1}\omega_{f1}$	1.0547	6.2	0.071446	0.0	0.73787	0.0
$1/T_\alpha$	66.195	0.4	−137.06	−0.1	161.51	0.0
$2\zeta_\alpha\omega_\alpha$	−0.29962	3.3	0.21941	0.3	0.77025	0.1
ω_α^2	23.886	0.6	83.063	0.1	269.75	0.0
$1/T_\theta$	0.33865	2.3	0.2387	−0.5	0.1013	0.0
$2\zeta_\theta\omega_\theta$	0.02696	−535.1	0.07755	−7.4	0.7534	0.0
ω_θ^2	23.747	0.1	83.054	0.0	269.75	0.0
$2\zeta_{\eta_1}\omega_{\eta_1}$	0.68664	0.0	0.37942	0.0	10.419[a]	0.0
$\omega_{\eta_1}^2$	−2.8133	0.9	5.7946	0.0	−7219.2[a]	0.0

[a]Coefficient of s^2 is zero, i.e., $\eta_1(s)/\delta(s) = F_{1_\delta}(2\zeta_{\eta_1}\omega_{\eta_1}s+\omega_{\eta_1}^2)/[(s^2+2\zeta_{sp}\omega_{sp}s+\omega_{sp}^2)(s^2+2\zeta_{f_1}\omega_{f_1}s+\omega_{f_1}^2)]$.

Table B2 Values and accuracy: nominal literal approximations

	Flexible aircraft		Supersonic missile		Hypersonic vehicle	
Parameter	Value	Error (%)	Value	Error (%)	Value	Error (%)
$\bar\omega_{sp}^2$	3.758	139.2	7.148	2.6	−4.253	−24.9
$2\bar\zeta_{sp}\bar\omega_{sp}$	1.246	42.5	0.3392	0.6	0.111	−22.4
$\bar\omega_{f1}^2$	34.83	−4.6	73.65	0.3	269.7	−0.5
$2\bar\zeta_{f1}\bar\omega_{f1}$	0.6214	−62.6	0.0696	−2.6	0.7703	4.3
$1/\bar T_\alpha$	66.195	0.4	−137.06	−0.1	161.51	0.0
$2\bar\zeta_\alpha\bar\omega_\alpha$	0.6214	−300.5	0.0696	−68.2	0.77025	−0.1
$\bar\omega_\alpha^2$	34.83	46.7	73.65	−11.3	269.75	0.0
$1/\bar T_\theta$	0.36358	9.8	0.2501	4.2	0.08445	−16.6
$2\bar\zeta_\theta\bar\omega_\theta$	0.6214	−10,126	0.0696	−16.9	0.7702	2.2
$\bar\omega_\theta^2$	34.83	46.9	73.65	−11.3	269.75	0.0

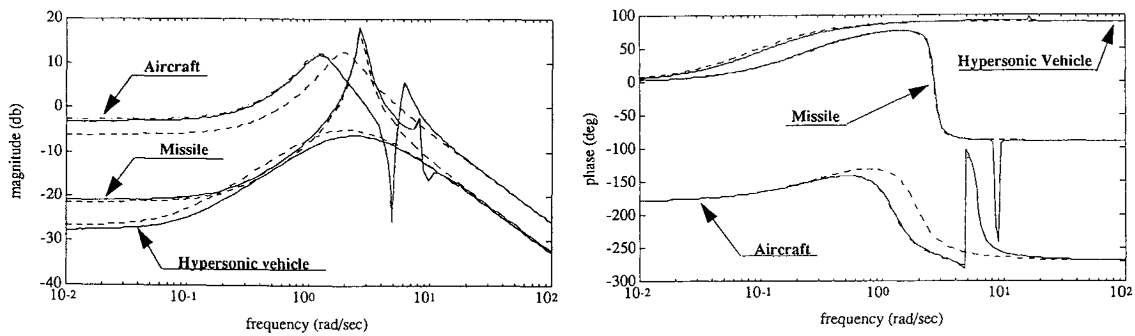

Fig. C2 $q(s)/\delta(s)$ [$= s\theta(s)/\delta(s)$] **frequency response for approximate literal models: ——, exact elastic short period model; - - - - -, nominal literal approximation; · · · · · ·, unified literal approximation.**

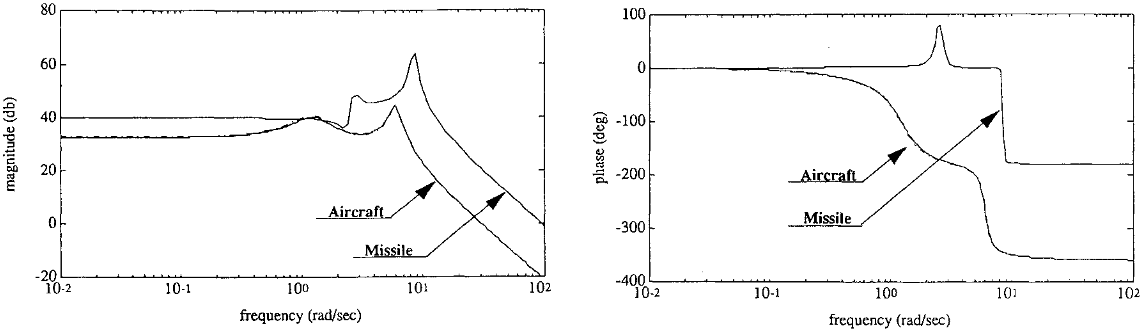

Fig. C3 $\eta_1(s)/\delta(s)$ **frequency response for approximate literal models: ——, exact elastic short period model; · · · · · ·, unified literal approximation.**

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